PRE-FILTERING FOR THE INITIALIZATION OF MULTI-WAVELET TRANSFORMS

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ABSTRACT

We introduce a new method for initializing the multi-wavelet decomposition algorithm. The approach assumes that the input signal is contained within some well-defined subspace of L_2 (e.g. space of bandlimited functions). The initialization algorithm is the orthogonal projection of the input signal into the space defined by the multi-scaling function. Unlike an interpolation approach, the projection method will always have a solution. We provide examples and implementation details.

1. INTRODUCTION

A multi-wavelet algorithm provides flexibility in terms of wavelet design over a traditional uni-wavelet algorithm (you can have compactness, orthogonality, symmetry, and regularity for example) in addition to providing superior energy compaction [1]. Mathematically, the pre-filtering problem can be expressed as follows. Let the vector

$$\Phi = \left(\varphi^1(x), \varphi^2(x), \varphi^3(x), ..., \varphi^r(x)\right),$$

be the multi-scaling function that generates the multi-resolution spaces

$$V_j = \left\{\sum_{i=1}^{r}\sum_{k\in Z} c_j^i(k) 2^{-\frac{j}{2}} \varphi^i\left(\frac{x}{2^j} - k\right); c_j^i \in l_2, i = 1, ..., r\right\}.$$

The multi-wavelet transform decomposes a signal $g(x) \in V_0$ as

$$g(x) = \sum_{i=1}^{r}\sum_{k\in Z} c_0^i(k)\varphi^i(x-k)$$

$$= \sum_{i=1}^{r}\sum_{k\in Z} c_{J_0}^i(k) 2^{-\frac{J_0}{2}} \varphi^i\left(\frac{x}{2^{J_0}} - k\right) + \sum_{i=1}^{r}\sum_{J_0\leq j<0}\sum_{k\in Z} d_j^i(k) 2^{-\frac{j}{2}} \psi^i\left(\frac{x}{2^j} - k\right)$$

where $\Psi = \left(\psi^1(x), \psi^2(x), \psi^3(x), ..., \psi^r(x)\right)$ is the multi-wavelet associated with the multi-resolution spaces V_j [1-5]. The multi-wavelet algorithm requires as input the coefficients $c_0^i(k)$ $i = 1, ..., r$. In practice, we have available only the samples $f[k/r]$, e.g. a digital image. The initialization problem (pre-filtering) involves determining the coefficients $c_0^i(k)$ from the samples $f[k/r]$. If we assume that $f(x) \in V_0$, then the problem can be posed as an interpolation problem. Otherwise, a more general approach is necessary. Xia et. al. [1] provides details on what to do when $f(x) \in V_0$. Here we provide a solution to the problem of pre-filtering when $f(x) \notin V_0$. The case of $f(x) \notin V_0$ is more realistic since we have only samples of $f(x)$ which are usually not related to the chosen multi-wavelet transform. These samples are however related to the impulse response of the acquisition device, which can provide a model for the signal.

Our pre-filtering operation is simply the orthogonal projection of the signal into the space defined by the multi-scaling function. The attractive features of this approach are that a solution always exists (unlike the interpolation approach described in [1]) and that we obtain the exact signal if it is already contained within V_0 (as is assumed in [1]).

2. PREVIOUS WORK

Assume that samples of the signal $f(x)$ are available. If the signal $f(x)$ is contained in V_0, then the initialization problem can be considered a problem of interpolating the function

$$f(x) = \sum_{i=1}^{r}\sum_{k\in Z} c_0^i(k)\varphi^i(x-k). \tag{1}$$

Since it is necessary to compute r coefficients for each knot, $f(x)$ must be sampled at a rate of $1/r$, (i.e. $f[k/r]$). Expressing (1) in terms of the samples of $f(x)$ provides a system of convolution equations which can be expressed in terms of the following matrix convolution

$$\mathbf{s}(k) = (\mathbf{B}*\mathbf{c})(k), \tag{2}$$

where the elements of the vectors and matrix are the following sequences:

$$[\mathbf{s}]_i(k) = f(k + (i-1)/r),$$
$$[\mathbf{B}]_{ij}(k) = \varphi^j(k - (i-1)/r),$$
$$[\mathbf{c}]_i(k) = c_0^i(k),$$

and the matrix convolution operator is defined as

$$[\mathbf{s}]_i(k) := \sum_{l=1}^{l=r}\sum_{h\in Z} [\mathbf{B}]_{il}(h)[\mathbf{c}]_l(k-h).$$

The problem is to find the coefficients $c_0^i(k)$ which involves the inversion (in the convolutional sense) of the matrix-sequence $\mathbf{B}(k)$. A drawback is that this inverse is not guaranteed to exist (an example will be shown later). Our approach of finding the coefficients $c_0^i(k)$, which we introduce next, does not suffer from this problem.

3. ORTHOGONAL PROJECTION METHOD

Again we assume that we have available samples, $f[k/r]$, of the signal $f(x)$. Unlike the interpolation approach however, we may assume that $f(x)$ does not belong to V_0, but is contained in some other subspace $S(\lambda_{1/r})$. This subspace should in principal depend on the impulse response of the device used to sample $f(x)$. For example, $f(x)$ may be bandlimited and then sampled, in which case $\lambda = \text{sinc}$ would be appropriate [6]. This subspace $S(\lambda_{1/r})$ must have the property that an interpolation basis exists. If the space does not have this property, then without additional information about the signal, it is not possible to obtain the necessary coefficients by linear filtering. The interpolation property of the space $S(\lambda_{1/r})$ will allow us to obtain from samples of $f(x)$ the coefficients $e(k)$ associated with the decomposition [7]

$$f(x) = \sum_{k \in Z} e(k)\lambda(xr - k),$$

where here we assume that $S(\lambda_{1/r})$ is defined by a single generating function $\lambda(x)$. Since we want to decompose the signal in terms of Φ, we would like to find the signal in V_0 that best approximates $f(x)$. If our criterion is least squares, then the solution is the orthogonal projection of $f(x)$ into V_0.

The approximation that we wish to find is given by

$$f_a(x) = \sum_{i=1}^{r} \sum_{k \in Z} c_0^i(k)\varphi^i(x - k).$$

The actual signal can be expressed as

$$f(x) = \sum_{i=1}^{r} \sum_{k \in Z} \left[(e*\delta_{i-1})\right]_{\downarrow r}(k)\lambda(rx - rk - (i-1))$$

$$= \sum_{i=1}^{r} \sum_{k \in Z} \left[(e*\delta_{i-1})\right]_{\downarrow r}(k)\lambda_{1/r}\left(x - k - (i-1)/r\right),$$

where $[b]_{\downarrow r}(k) = b(kr)$, $\lambda_{1/r}(x) = \lambda(rx)$, and $\delta_i(k)$ denotes the unit impulse sequence located at $k = i$.

The approximation $f_a(x)$ (or equivalently $c_0^i(k)$ $i = 1,...,r$) is found by solving the orthogonality condition

$$\langle f(x) - f_a(x), \varphi^i(x - k)\rangle = 0 \qquad i = 1,...,r, \quad k \in Z.$$

These equations can be expressed in the following matrix convolution format

$$(A*c)(k) = (D*e)(k)$$

where

$$[A]_{ij}(k) = \left(\varphi^j * \varphi^{iv}\right)(k);$$
$$[c]_i(k) = c_0^i(k);$$
$$[D]_{ij}(k) = \left(\lambda_{1/r}^j * \varphi^{iv}\right)(k);$$
$$[e]_i(k) = \left[(e*\delta_i)\right]_{\downarrow r}(k);$$
$$\varphi^{iv}(x) = \varphi^i(-x);$$
$$\lambda_{1/r}^j(x) = \lambda_{1/r}\left(x - (j-1)/r\right).$$

The coefficients $c_0^i(k)$ are obtained by inverting the matrix-sequence $A(k)$. Unlike the interpolation approach, the convolution-inverse of $A(k)$ is guaranteed to exist since Φ is a Riesz basis of V_0 [8]. Thus, the orthogonal projection is

$$c(k) = \left((A)^{-1} * D * e\right)(k).$$

A system diagram is shown in Figure 1. The filter $\left(\lambda_{1/r}\right)^{-1}(k)$ is the convolution inverse of the discrete filter $\lambda_{1/r}(k)$ [7]. In the next section, we provide an example implementation.

4. IMPLEMENTATION AND RESULTS

To demonstrate the usefulness of our method, we consider the initialization of a multi-wavelet transform associated with the Hermite cubic spline multi-resolution [9]. The two chosen $(r = 2)$ scaling functions are shown in Figures 2 and 3. These
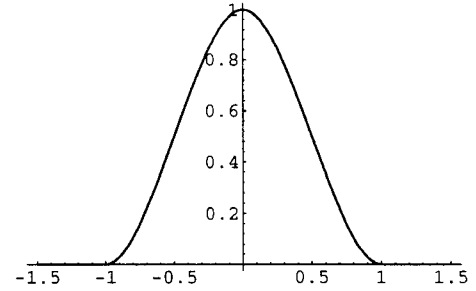
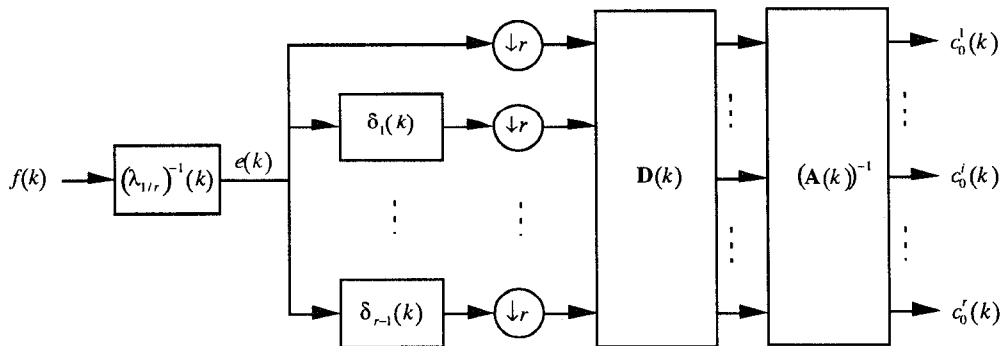Figure 2: $\varphi^1(x)$ of the Hermite cubic spline multi-resolution.

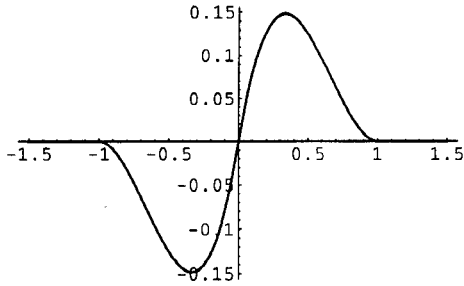Figure 1: System diagram of orthogonal projection initialization.

Figure 3: $\varphi^2(x)$ of the Hermite cubic spline multi-resolution.

functions have the property that if

$$f(x) = \sum_{k \in Z} c_0^1(k)\varphi^1(x-k) + c_0^2(k)\varphi^2(x-k),$$

then the coefficients are given by $c_0^1(k) = f(x)\big|_{x=k}$ and $c_0^2(k) = f'(x)\big|_{x=k}$.

4.1 Interpolation Approach

For the Hermite cubic spline multi-scaling function, an interpolating set of functions does not exist. In other words, the convolutional inverse of matrix $\mathbf{B}(k)$ (c.f., eq. (2)) does not exist. The singularity can be easily shown by computing the Fourier transform of the filter

$$\mathbf{B}(k) = \begin{bmatrix} (1,0) & (0,0) \\ (1/2,1/2) & (-1/8,1/8) \end{bmatrix},$$

which is given by the matrix

$$\hat{\mathbf{B}}(f) = \begin{bmatrix} 1 & 0 \\ (1/2)(1+e^{-j2\pi f}) & (1/8)(e^{-j2\pi f}-1) \end{bmatrix}.$$

For $(\mathbf{B})^{-1}(k)$ to exist, the matrix $\hat{\mathbf{B}}(f)$ must be nonsingular for $f \in [0,1]$, which is not true since $\hat{\mathbf{B}}(0)$ is singular.

4.2 Orthogonal Projection Approach

If the signal is known exactly, or if we have samples of the signal and it is known to be contained in some space for which an interpolating basis does exist, then we can always apply our projection initialization. If we are operating with samples of the signal, then the space that we use to interpolate $f(x)$ can, but need not be within V_0. If the signal is within V_0, then the approximation will be exact since we are performing a projection operation.

Example 1: Input signal not in V_0

As an example let us assume that our signal is contained within the space $S(\beta_{1/2})$ defined by a 1/2 scaled cubic B-spline function. That is

$$f(x) = \sum_{k \in Z} e(k)\beta(2x-k).$$

The coefficients $e(k)$ are easily computed from the samples $f(k/2)$ by use of the inverse filter $(\beta_{1/2})^{-1}(k)$[7]. Note that

since $f(x)$ is not guaranteed to have C^{∞} smoothness at the points $x = (2k+1)/2$, the space $S(\beta_{1/2})$ is not contained in the Hermite cubic spline space V_0. The signal shown in Figure 4 is contained in $S(\beta_{1/2})$ but not V_0. Figure 7 displays the orthogonal projection of this signal into the space V_0. The coefficients $c_0^1(k)$ and $c_0^2(k)$ shown in Figures 5 and 6 are the sample values of the projection and the sample values of the projection's derivative respectively.
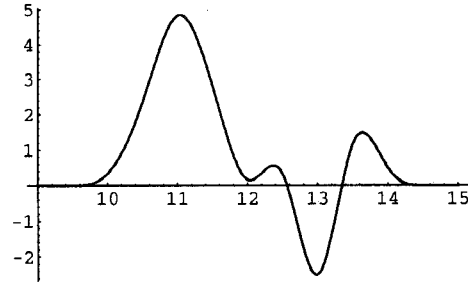

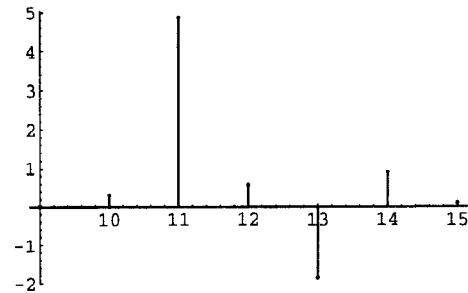Figure 4: A signal contained in $S(\beta_{1/2})$ but not V_0.

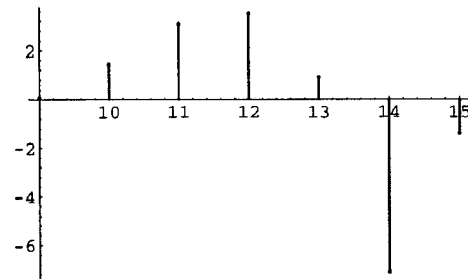Figure 5: Computed $c_0^1(k)$ coefficients for the signal in Fig. 4.

Figure 6: Computed $c_0^2(k)$ coefficients for the signal in Fig. 4.
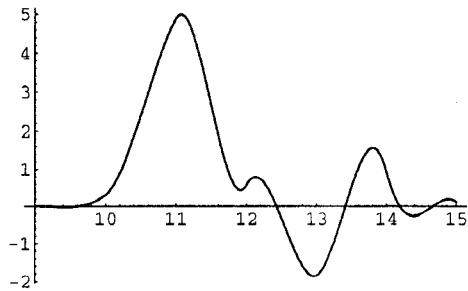
Figure 7: Orthogonal projection of the signal in Fig. 4 into V_0.

Figure 10: Computed $c_0^2(k)$ coefficients for the signal in Fig. 8.

Example 2: Input signal in V_0

For this example we use a function in the cubic B-spline space with integer knots, $S(\beta)$. The signal has the form

$$f(x) = \sum_{k \in Z} e(k)\beta(x - k).$$

It is not difficult to show that $S(\beta) \subset V_0$. Filtering this signal through the system in Figure 1 provides the coefficients displayed in Figures 9 and 10. Since the signal is already in the space V_0 our projection approach provides coefficients that are exactly the sample values of the signal and its derivative. For this example, the direct interpolation approach [1] cannot obtain $c_0^1(k)$ and $c_0^2(k)$ even though the signal is in V_0. The reason is that the matrix-sequence $(\mathbf{B})^{-1}(k)$ does not exist as mentioned at the beginning of this section.
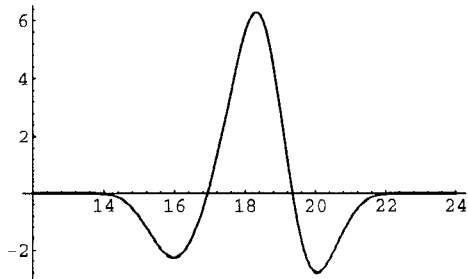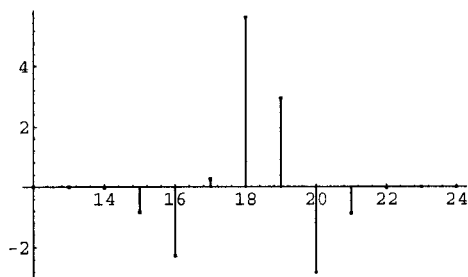

Figure 8: A signal contained in $S(\beta)$ and V_0.

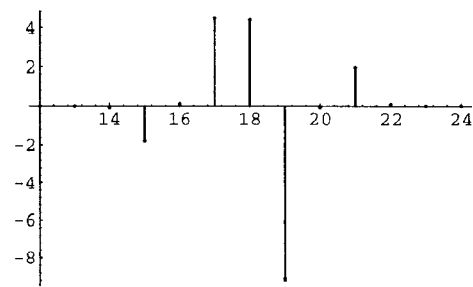Figure 9: Computed $c_0^1(k)$ coefficients for the signal in Fig. 8.

5. CONCLUSION

We have introduced a new method for initializing (pre-filtering) the multi-wavelet decomposition algorithm. The approach has the following attractive features:

- The projection filter will always exist unlike the interpolation filter.

- If the signal is contained within the space defined by the multi-scaling function, then the projection solution will be an exact approximation of the signal.

- The method is more flexible and general than previous methods since it provides a least squares solution when the original signal is not contained in the space defined by the multi-scaling function.

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