# The Mathematics of Color Calibration

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**Abstract**

*The mathematical formulation of calibrating color image reproduction and recording devices is presented. This formulation provides a foundation for future research in areas of characterization of devices and display of color images. The procedure outlined in this paper should become standard for displaying color images for the image processing community.*

## 1 Introduction

With the advent of low cost color printers and scanners, there is increased interest in the image processing community to develop techniques for enhancing, restoring, and reproducing color images [1]. The demonstration of the performance of methods for color image processing has presented a problem due to an inability to control the final printing process, misunderstandings with regard to color spaces and what RGB really means, and improper or poorly managed scanning to name a few. Problems can be clearly seen in comparing the various "unprocessed" color images of Lena in [1] (a recent special issue on color imaging) with the actual original Lena image. A simulation of the differences is available at the web site ftp://ftp.eos.ncsu/pub/hjt/profile. The difference between the images demonstrates the need for a standard image defined in terms of CIE values with which to demonstrate color image processing algorithms, in addition to standards for displaying and porting processed images.

## 2 Background

We will use a vector space notation in which the visible spectrum is mathematically sampled at $N$ wavelengths. An illuminant spectrum is represented by an $N \times N$ diagonal matrix and the spectral reflectance of an object is represented by an $N$ element vector $\mathbf{r}$. The radiant spectrum reflected from the object with spectral reflectance $\mathbf{r}$ under the illuminant $\mathbf{L}$ can be expressed as $\mathbf{Lr}$. The columns of the $N$x3 matrix $\mathbf{A}$ contain the sampled CIE XYZ color matching functions and the CIE XYZ value of the spectrum $\mathbf{Lr}$ is given by $\mathbf{t} = \mathbf{A}^T\mathbf{Lr}$.

A *device independent color space* is defined as any space that has a one-to-one mapping onto the CIE XYZ color space. Device independent values describe color for the standard CIE observer.

By definition, a *device dependent color space* cannot have a one-to-one mapping onto the CIE XYZ color space. In the case of a recording device, the device dependent values describe the response of that particular device to color. For a reproduction device, the device dependent values describe only those colors the device can produce.

The use of device dependent descriptions of color presents a problem in the world of networked computers and printers. The same RGB vector can result in different colors on different monitors and printers. Similarly, a specified CMYK value can result in different colors on different printers.

A solution is to define images in terms of a CIE color space and then transform this data to device dependent descriptors for the device on which the image is to be reproduced. This solution requires knowledge of a function $\mathcal{F}_{device}$ which will provide a mapping from device dependent control values to a CIE color space. In the case of a printer, it is necessary to determine a transformation $\mathcal{F}^{-1}_{device}$ (which may or may not exist). For a monitor, the transformations $\mathcal{F}_{device}$ and $\mathcal{F}^{-1}_{device}$ are both needed since the monitor is used as both a source of image data, and as a display device.

Modern printers and display devices are limited in the colors they can produce. This limited set of colors is defined as the *gamut* of the device. If $\Omega_{cie}$ is the range of numerical values in the selected CIE color space and $\Omega_{print}$ is the numerical range of the device control values then the set

$$G = \{ \ \mathbf{t} \in \Omega_{cie} \mid \exists \ \mathbf{c} \in \Omega_{print} \text{ where } \mathcal{F}_{device}(\mathbf{c}) = \mathbf{t} \ \}$$

defines the gamut of the color output device. Similarly, the complement set

$$G^c = \{\ \mathbf{t} \in \Omega_{cie} \mid\ \nexists\ \mathbf{c} \in \Omega_{print} \text{ where } \mathcal{F}_{device}(\mathbf{c}) = \mathbf{t}\ \}$$

defines colors outside the device gamut. For colors in the gamut, there will exist a mapping between the device dependent control values and the CIE XYZ color space. Colors which are in $G^c$ cannot be reproduced and must be *gamut-mapped* to a color which is within $G$. The gamut mapping algorithm $\mathcal{D}$ is a mapping from $\Omega_{cie}$ to $G$, that is $\mathcal{D}(\mathbf{t}) \in G \quad \forall \mathbf{t} \in \Omega_{cie}$.

The mappings $\mathcal{F}_{device}$, $\mathcal{F}_{device}^{-1}$, and $\mathcal{D}$ make up what is defined as a *device profile*. These mappings describe how to transform between a CIE color space and the device control values. The International Color Commission (ICC) has suggested a standard format for describing a profile[7].

## 3 Monitors

A monitor is often used to provide softcopy preview for the printing process and is a common source for user generated images. Monitor calibration is almost always based on a physical model of the device [5]. A typical model is $\mathbf{t} = \mathbf{H}[r', g', b']^T$ where

$$\begin{aligned} r' &= [(r - r_0)/(r_{max} - r_0)]^{\gamma_r} \\ g' &= [(g - g_0)/(g_{max} - g_0)]^{\gamma_g} \\ b' &= [(b - b_0)/(b_{max} - b_0)]^{\gamma_b} \end{aligned} \tag{1}$$

where $\mathbf{t}$ is the CIE value produced by driving the monitor with control value $\mathbf{d} = [r, g, b]^T$, and the parameters $\gamma_r$, $\gamma_g$, $\gamma_b$, $r_0$, $g_0$, $b_0$, $r_{max}$, $g_{max}$, $b_{max}$, and $\mathbf{H}$ are defined in the profile.

Creating a profile for a monitor involves the determination of these parameters where $r_{max}$, $g_{max}$, $b_{max}$ are the maximum values of the control values (e.g. 255). To determine the parameters, a series of color patches is displayed on the monitor and measured with a colorimeter which will provide pairs of CIE values $\{\mathbf{t}_k\}$ and control values $\{\mathbf{d}_k\}$ $k = 1, ..., M$.

Values for $\gamma_r$, $\gamma_g$, $\gamma_b$, $r_0$, $g_0$, and $b_0$ are determined such that the elements of $[r', g', b']$ are linear with respect to the elements of $XYZ$ and scaled between the range [0,1] (cf. Eq. 1). The matrix $\mathbf{H}$ is determined using

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_{Rmax} & X_{Gmax} & X_{Bmax} \\ Y_{Rmax} & Y_{Gmax} & Y_{Bmax} \\ Z_{Rmax} & Z_{Gmax} & Z_{Bmax} \end{bmatrix} \begin{bmatrix} r' \\ b' \\ g' \end{bmatrix}$$

where $[X_{Rmax}, Y_{Rmax}, Z_{Rmax}]^T$ is the CIE XYZ tristimulus value of the red phosphor for control value $\mathbf{d} = [r_{max}, 0, 0]^T$. The other phosphors are defined similarly.

## 4 Scanners

Mathematically, the recording process of a scanner can be expressed as

$$\mathbf{z}_i = \mathcal{H}(\mathbf{M}^T \mathbf{r}_i)$$

where the matrix $\mathbf{M}$ contains the spectral sensitivity (including the scanner illuminant) of the three (or more) bands of the device, $\mathbf{r}_i$ is the spectral reflectance at spatial point $i$, $\mathcal{H}$ models any nonlinearities in the scanner (invertible in the range of interest), and $\mathbf{z}_i$ is the vector of recorded values.

The calibration problem is to determine a continuous mapping $\mathcal{F}_{scan}$ which will transform the recorded values to a CIE color space, i.e.

$$\mathbf{t} = \mathbf{A}^T \mathbf{L} \mathbf{r} = \mathcal{F}_{scan}(\mathbf{z})$$

for all $\mathbf{r} \in \Omega_r$, the space of reflection values.

Look-up-tables, nonlinear and linear models for $\mathcal{F}_{scan}$ have been used to calibrate color scanners [3, 4]. In all of these approaches, the first step is to select a collection of color patches which span the colors of interest. Ideally these colors should not be metameric in terms of the scanner sensitivities or to the standard observer under the illuminant for which the calibration is being produced. This constraint is easily obtained and assures a one-to-one mapping between the scan values and the device independent values across these samples.

The reflectance spectra of these $M_q$ color patches will be denoted by $\{\mathbf{q}\}_k$ for $1 \leq k \leq M_q$ . These patches are measured using a spectrophotometer or a colorimeter which will provide the device independent values

$$\{\mathbf{t}_k = \mathbf{A}^T \mathbf{q}_k\} \quad \text{for} \quad 1 \leq k \leq M_q.$$

These values $\{\mathbf{t}_k\}$ could represent any colorimetric or device independent values, e.g. CIELAB, CIELUV in which case $\{\mathbf{t}_k = \mathcal{L}(\mathbf{A}^T \mathbf{q}_k)\}$ where $\mathcal{L}(\cdot)$ is the transformation from CIEXYZ to the appropriate color space. The patches are also measured with the scanner to be calibrated providing $\{\mathbf{z}_k = \mathcal{H}(\mathbf{M}^T \mathbf{q}_k)\}$ for $1 \leq k \leq M_q$.

Mathematically, the calibration problem is: find a transformation $\mathcal{F}_{scan}$ where

$$\mathcal{F}_{scan} = \arg(\min_{\mathcal{F}} \sum_{i=1}^{M_q} ||\mathcal{F}(\mathbf{z}_i) - \mathbf{t}_i||^2)$$

and $||.||^2$ is the error metric in the CIE color space. Other metrics may be used if desired. In practice, it may be necessary and desirable to incorporate constraints on $\mathcal{F}_{scan}$.

## 5 Printers

Printer calibration is difficult due to the nonlinearity of the printing process, and the wide variety of methods used for color printing (e.g. lithography, inkjet, dye sublimation etc.). Because of these difficulties, printing devices are often calibrated using an LUT and interpolation [2, 3].

To produce a profile of a printer, a subset of values spanning the space of allowable control values for the printer is first selected. Denote these device dependent values by $\mathbf{c}_k$ for $1 \leq k \leq M_p$. These values produce a set of reflectance spectra which are denoted by $\mathbf{p}_k$ for $1 \leq k \leq M_p$. The patches $\mathbf{p}_k$ are measured using a colorimetric device as was performed for the scanner calibration, which provides the values

$$\{\mathbf{t}_k = \mathbf{A}^T \mathbf{p}_k\} \quad \text{for} \quad 1 \leq k \leq M_p$$

Again, $\mathbf{t}_k$ could represent any colorimetric or device independent values, not just CIEXYZ.

The problem is then to determine a mapping $\mathcal{F}_{print}$ which is the solution to the optimization problem

$$\mathcal{F}_{print} = \arg(\min_{\mathcal{F}} \sum_{i=1}^{M_p} ||\mathcal{F}(\mathbf{c}_i) - \mathbf{t}_i||^2)$$

where as in the scanner calibration problem, there may be constraints which $\mathcal{F}_{print}$ must satisfy.

## 6 Gamut Mapping

Consider two gamuts $G_{monitor}$ and $G_{print}$. It is desired to print an image which is displayed on the monitor. Assuming $\mathcal{F}_{monitor}$ is known, we can map from the monitor RGB values to CIE values. Now the problem is to map these CIE values to device dependent values for the printer. The problem is that there may be colors which the monitor can display but the printer cannot print. This problem is defined as a gamut mismatch problem (the mismatch in this case being between the monitor and the printer). The mapping $\mathcal{D}$ is used for this purpose and the device control value for the CIE value $\mathbf{t}$ is given by $\mathcal{F}^{-1}_{printer}(\mathcal{D}(\mathbf{t}))$.

Depending upon the desired effect, the gamut mapping function $\mathcal{D}$ may or may not be the identity operator on the colors within $G_{print}$. The reason for using a $\mathcal{D}$ which is not the identity operator can be illustrated by an example in which there are smoothly varying regions in the image which are outside the device gamut. These colors will be gamut-mapped to the same color on the gamut boundary which will result in abrupt edges in the previously smooth region. To reduce this artifact, a gamut mapping is often used which compresses all the colors in the image to reduce the colorimetric dynamic range in the image while ensuring that all the colors can be reproduced. For detailed gamut mapping experiments, the reader is referred to [9, 8].

## 7 Constraints

The formulation for the scanner calibration and the printer calibration are mathematically similar. In many practical cases, there are constraints which the mappings should satisfy. For example, in the printer profile there are several factors such as ink limit and undercolor removal which often come into play.

In the case of a printer profile, constraint sets of interest include the following:

data consistency:

$$\mathcal{F}_{print} \in \{ \ \mathcal{G} \mid ||\mathcal{G}(\mathbf{c}_i) - \mathbf{t}_i|| \leq \delta_v \quad i = 1, ..., M_p \ \}$$

where $\delta_v$ is a just-noticeable-difference (JND) threshold.

inklimit:

$$\mathcal{F}_{print} \in \{ \ \mathcal{G} \mid ||\mathcal{G}^{-1}(\mathbf{t})|| \leq \delta_{ink} \ \ \forall \mathbf{t} \in G \ \}$$

where $\delta_{ink}$ is the maximum amount of ink that should be placed on the paper.

smoothness:

$$\mathcal{F}_{print} \in \{ \ \mathcal{G} \mid ||(\nabla \mathcal{G})(\mathbf{c})|| \leq \delta_{smooth} \ \ \forall \mathbf{c} \in \Omega_{print} \ \}$$

where $\nabla \mathcal{G}$ is the gradient of the function $\mathcal{G}$ and $\Omega_{print}$ is the range of control values for the printer.

In CMYK printers, under-color removal is a technique in which the cyan, magenta and yellow ink amounts are reduced and black ink is added. Typically, the allowable CMYK values are restricted since different CMYK combinations could give rise to the same CIE XYZ value. This restriction is defined by a set $H$ which is itself defined by four curves which map values from $[cmy]$ 3-space into $[c'm'y'k']$ four space. For example, $H$ could be defined as

$$H = \{ \ [c+g_c(\alpha), m+g_m(\alpha), y+g_y(\alpha), f_k(\alpha)] \mid \alpha = \min(c, m, y) \ \}$$

where $c, m, y \in [0, 255]$, $g_i(\alpha) = f_i(\alpha) - \alpha$, and the curves $f_i(\alpha)$, for $i = (c, m, y, k)$ are as shown in Figure 1. With $H$ defined, the constraint of under-color removal is given by:

$$\mathcal{F}_{print} \in \{ \ \mathcal{G} \mid \mathcal{G}^{-1}(\mathbf{t}) \in H \ \ \forall \mathbf{t} \in G \ \}$$

In the LUT based profile, the gamut mapping is implicit in the function which maps from the CIE color space to the device dependent values. In constructing the LUT, gamut-mapping considerations can be incorporated as constraints on the LUT entries. For

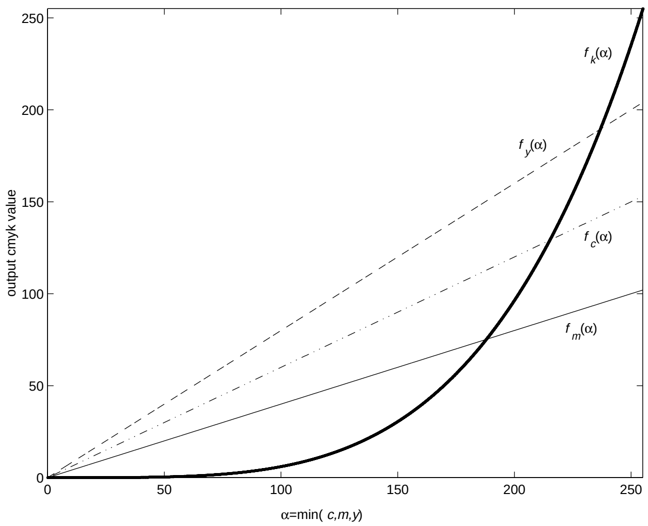


Figure 1: Undercolor removal curves

example, it may be desired that certain points outside the gamut in the LUT provide maximum colorant on the paper (e.g. 100% cyan, 0% yellow,0% magenta, 0% black). At the same time one would like to have a smooth transition to the in-gamut colors, and a colorimetrically meaningful way to assign the out-of-gamut colors. Constraints of interest in determining the function $\mathcal{D}$ include minimum color error:

$$\begin{aligned}\mathcal{D} \quad \in \quad & \{\ \mathcal{G} \mid \mathcal{G}(\mathbf{t}) \in G \quad \mathcal{G} = \arg(\min_{\mathcal{K}} ||\mathcal{P}(\mathcal{K}(\mathbf{t})) - \mathcal{P}(\mathbf{t})||^2) \\ & \forall \mathbf{t} \in \Omega_{CIE}\ \}\end{aligned}$$

where $\mathcal{P}$ is a mapping from the CIE color space to a space which is perceptually optimal for gamut mapping (e.g. constant hue)

smoothness:

$$\mathcal{D} \in \{\ \mathcal{G} \mid ||(\nabla\mathcal{G})(\mathbf{t})|| \leq \delta \ \ \forall \mathbf{t} \in \Omega_{CIE}\ \}$$

fixed points:

$$\mathcal{D} \in \{\ \mathcal{G} \mid \mathcal{G}(\mathbf{t}_i) = \mathbf{s}_i \quad i = 1, ..., P\}$$

where there are $P$ fixed points, $\{\mathbf{t}_i\}$, $\{\mathbf{s}_i\}$ $i = 1, ..., P$, which need to be mapped exactly.

Scanner calibration can also be formulated in this framework. The primary sets of interest include

data consistency:

$$\mathcal{F}_{scan} \in \{\ \mathcal{G} \mid ||\mathcal{G}(\mathbf{u}_i) - \mathbf{t}_i|| \leq \delta_v \quad i = 1, ..., M_p\ \}$$

smoothness:

$$\mathcal{F}_{scan} \in \{\ \mathcal{G} \mid ||(\nabla\mathcal{G})(\mathbf{u})|| \leq \delta \ \ \forall \mathbf{u} \in \Omega_{scan}\ \}$$

where $\Omega_{scan}$ is the range of numerical values produced by the scanner.

Finally, color perception can be included into the construction of the profile by performing the transformation to the perceptual color space on the measurements from the samples which are used for the construction of the profile. If $\mathcal{L}$ maps the CIE values to a perceptual color space such as that found in [6], then the profile LUT is constructed between the values $\{\mathbf{v}_k = \mathcal{L}(\mathbf{t}_k)\}$ and the device control values $\mathbf{c}_k$.

## 8 Example

To demonstrate the need for improved color management, an original version of the Lena image was scanned on a desktop scanner. The scanned image will be referred to as the newly scanned Lena image. In addition, a color target was scanned which contained 276 color patches. The target patches were measured with a colorimeter and an LUT mapping was determined from scanned RGB values to CIELab D50 (i.e. $\mathcal{F}_{scan}$ was created). Finally, the commonly used digital color Lena image was obtained which will be referred to as the standard Lena image.

A dye sublimation (dye-sub) RGB printer (3-color CMY printer which accepted RGB input values) was also profiled from CIELab D50 (i.e. $\mathcal{F}^{-1}_{print}$ was created) to printer RGB space. As discussed in Section 5, this profile was created by measuring a series of color patches with a colorimeter and creating an LUT. Out of gamut colors were mapped, via $\mathcal{D}$, to the closest in gamut value, along a constant hue angle while preserving lightness in CIELab space.

The LUT for the dye-sub printer was used to map from the CIELab D50 values to printer control values. Three images were printed and are available at ftp://ftp.ncsu.edu/pub/hjt/profile. The images are as follows where $\mathbf{l}_k$ is the $k$th pixel RGB value for the standard Lena image and $\mathbf{ls}_k$ is the $k$th pixel RGB value for the newly scanned Lena image:

(a) The standard Lena image, created by sending the RGB values $\mathbf{l}_k$ to the printer.

(b) Image created by sending the values $\mathcal{F}^{-1}_{print}(\mathcal{D}((\mathcal{F}_{scan}(\mathbf{ls}_k)))$ to the printer.

(c) Image created by sending the values $\mathbf{ls}_k$ to the printer.

Figure 2 demonstrates the differences between (a), (b), and (c) by separately displaying the CIELab frames (i.e. individual $L^*$, $a^*$, and $b^*$ images) for each case. The images were created by applying the transformation $\mathcal{F}_{print}$ on the printer RGB control values. For this figure the $L^*$, $a^*$, and $b^*$ values were uniformly quantized over their physical range with 8bits. While not as effective as displaying the color images directly,

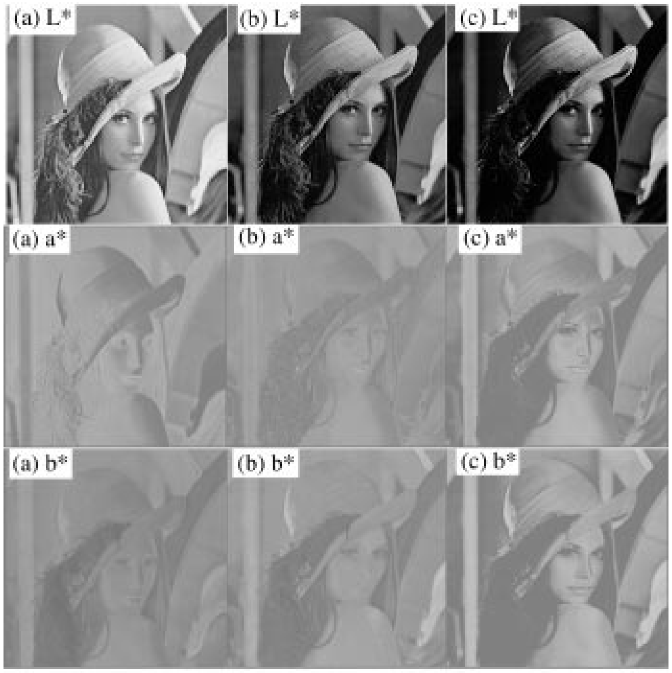


Figure 2: CIELab frames for cases (a), (b) and (c)

the figure does demonstrate the large differences between cases (a), (b) and (c).

When viewed as a color image, case (a) closely matched those in the special issue, while (b) was a close match to the original printed image. Providing the scanned RGB values directly to the printer (case (c)) produces a much different image. One should realize that the commonly used standard Lena image is a scanned RGB image which is often provided directly as input to the available color printer. Comparison of (a) and (c) gives an indication of the differences in the raw scanner data. The large differences between the images demonstrate that widely different results can occur depending upon how an image is scanned, processed, and printed. This is a significant problem when attempting to convey the results of a color image processing algorithm, and to compare those results with previously archived material.

## 9 Conclusions

The problem of achieving color management by calibrating scanners, monitors, and printers was defined mathematically. This lays a framework upon which sophisticated color image processing methods may be developed. The ease with which the constraint sets are formulated, as shown in Section 7, demonstrates the usefulness of the framework. The importance of the problem was made clear by consideration of the vast differences in appearance between images produced by calibrated and uncalibrated systems. This more formal approach of producing calibrated color output should become standard procedure when displaying images for the image processing community.

## References


[1] H. J. Trussell, J. Allebach, M. D. Fairchild, B. Funt, and P. W. Wong Eds., Special Issue on Color Imaging, *IEEE Transactions on Image Processing*, Vol. 6, No. 7, July 1997.

[2] J. Z. Chang, J. P. Allebach, and C. A. Bouman, “Sequential linear interpolation of multidimensional functions,” *IEEE Transactions on Image Processing*, Vol. 6, No. 9, pp. 1231-1245, Sept. 1997.

[3] P. C. Hung, “Colorimetric calibration in electronic imaging devices using a look-up table model and interpolations,” *J. Electronic Imaging*, Vol. 2, pp. 53-61, Jan. 1993.

[4] H. R Kang and P. G. Anderson, “Neural network applications to the color scanner and printer calibrations,” *J. Electronic Imaging*, Vol. 1, pp. 125-134, April 1992.

[5] W. B. Cowan, “An inexpensive scheme for calibration of a color monitor in terms of standard CIE coordinates,” *Comput. Graph.*, vol. 17, pp. 315-321, July 1983.

[6] M. D. Fairchild, “Refinement of the RLAB color space,” *Color Res. Appl.*, vol. 21, pp. 338-346, Oct. 1996.

[7] International Color Consortium, *Int. Color Consort. Profile Format Ver. 3.4*, available at http://color.org/.

[8] T. Hoshino and R. S. Berns, “Color Gamut Mapping Techniques for Color Hard Copy Images,” in *Proc. SPIE,* Vol. 1990, pp. 152-165, 1993.

[9] E. D. Montag and M. D. Fairchild, “Psychophysical Evaluation of Gamut Mapping Techniques Using Simple Rendered Images and Artificial Gamut Boundaries,” *IEEE Trans. Image Proc.*, vol. 6, no. 7, pp. 977-989, July 1997.